Small time control of bilinear Schrödinger equations

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Joint work with
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The bilinear Schrödinger equation

$$i\partial_t\psi(t,x) = \Big(-\Delta + V(x) + \sum_{j=1}^{m} u_j(t) W_j(x) \Big)\psi(t,x)\,, \qquad x \in M\,.$$

where M boundaryless manifold, and $V, W_j : M \to \mathbb{R}$.

<u>*Control*</u> : $u = (u_1, \dots, u_m) : (0,T) \to \mathbb{R}^m$
<u>*State*</u> : $\psi(t) \in \mathcal{S}$, the unit $L^2(M,\mathbb{C})$-sphere
<u>*Question*</u> : **Small-Time Global Controllability (STGC)**
$\forall T > 0, \psi_0, \psi_* \in \mathcal{S}, \quad \exists u : (0,T) \to \mathbb{R}^m \quad$ s.t. $\quad \psi(T; u, \psi_0) = \psi_*$

- Model valid only on **short time scales**.
- Global constant phases are ignored ($W_m = 1$).

Goal : Identify frameworks M, V, W_j for which STGC holds.

We will prove it in the 2 cases :

- $M = \mathbb{T}^d$, $m = 2d+1$, $W_{2j-1} = \sin(x_j)$, $W_{2j} = \cos(x_j)$
- $M = \mathbb{R}^d$, $m = d+2$, $W_j = x_j$, $W_{d+1} = e^{-|x|^2}$

Toy model in finite dimension

$$x'(t) = \left(A + \sum_{j=1}^{m} u_j(t) B_j\right) x(t)$$

$A, B_j \in \mathfrak{u}(n)$: skew-hermitian
$x(t) \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$

- **Small-Time Global Controllability in $\mathbb{S}^{2n-1}$:**

$\forall T > 0, x_0, x_* \in \mathbb{S}^{2n-1}, \quad \exists u \in L^1(0,T) \quad$ s.t. $\quad x(T; u, x_0) = x_*$

Another control pb can be expressed via the resolvent matrix :

$$\begin{cases} R'(t) = \left(A + \sum_{j=1}^{m} u_j(t) B_j\right) R(t) \\ R(0) = I_n \end{cases}$$

$R(t) \in U(n)$
unitary matrix

- **Small-Time Global Controllability in $U(n)$:**

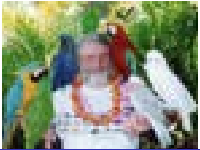
$\forall T > 0, R_* \in U(n), \quad \exists u \in L^1(0,T) \quad$ s.t. $\quad R(T; u) = R_*$

STGC in $U(n)$ $\quad \Longrightarrow \quad$ **STGC in $\mathbb{S}^{2n-1}$**

STGC in $\mathbb{S}^{2n-1}$ $\quad \Longleftrightarrow \quad$ $G := \bigcap_{T>0} R(T; L^1)$ acts transitively on $\mathbb{S}^{2n-1}$

$\forall x_0, x_* \in \mathbb{S}^{2n-1}, \exists R_* \in G$ s.t. $R_*(x_0) = x_*$

compact Lie groups
[Jurdjevic-Sussmann 1971]



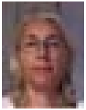
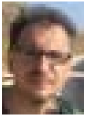
Toy model : Large Time Global Controllability

$$x'(t) = \left(A + \sum_{j=1}^m u_j(t) B_j\right) x(t)$$

$A, B_j \in \mathfrak{u}(n)$: skew-hermitian
$x(t) \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$

Large Time GC in $U(n)$ $\iff$ **Lie**$\{A, B_1, \dots, B_m\} = \mathfrak{u}(n)$
Large Time GC in $\mathbb{S}^{2n-1}$ $\iff$ **Lie**$\{A, B_1, \dots, B_m\} = \mathfrak{u}(n)$ **or** $\mathfrak{sp}(n/2)$

:0) computationally checkable NSC for global controllability :0)


[Albertini-D'Alessandro 2001, Brockett 1970]

For $\mathcal{F} \subset \mathcal{M}_n(\mathbb{C})$, Lie$\{\mathcal{F}\}$ is the smallest vector subspace of $\mathcal{M}_n(\mathbb{C})$ containing $\mathcal{F}$ and commutator-stable.
The commutator is defined by $\mathrm{ad}_A(B) = [A, B] = AB - BA$

Example of difficulty in small-time :

Take $x' = Ax + u(t)Bx$ and $B\varphi = \lambda\varphi$ thus $\lambda \in i\mathbb{R}$.
Then $\frac{d}{dt}\langle x(t), \varphi\rangle = \langle Ax(t), \varphi\rangle + u(t)\lambda\langle x(t), \varphi\rangle$ thus

$$\Big| |\langle x(t), \varphi\rangle| - |\langle x(0), \varphi\rangle| \Big| \leq \left| \langle x(t), \varphi\rangle - \langle x(0), \varphi\rangle e^{\lambda \int_0^t u} \right| \leq \|A\| t$$

$\longrightarrow$ no controllability in time $T < \frac{1}{\|A\|}$

Toy model : Small Time Global Controllability

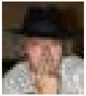
$$x'(t) = \left(A + \sum_{j=1}^{m} u_j(t) B_j\right) x(t)$$

$A, B_j \in \mathfrak{u}(n)$: skew-hermitian
$x(t) \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$

STGC in $\mathbb{S}^{2n-1} \iff$ **Lie**$\{B_1, \dots, B_m\} = \mathfrak{u}(n)$ **or** $\mathfrak{sp}(n/2)$

The short-time scale renders the drift A ineffective.

Proof : • $\underline{x}' = \sum u_j B_j \underline{x}$, $\quad x' = (A + \sum u_j B_j)x$, $\quad \underline{x}(0) = x(0)$
imply $\|(\underline{x} - x)(t)\| \leq \|A\| t$.
• If the Lie group generated by $\{B_1, \dots, B_m\}$ does not act transitively on $\mathbb{S}^{2n-1}$, then there exist at 2 orbits, at positive distance δ (compact). Thus, global control does not hold in time $T < \frac{\delta}{2\|A\|}$. $\square$

[Agrachev-Boscain-Gauthier-Sigalotti 2017]

A misleading intuition for PDEs : the $B_j = iW_j(x)$ commute and the unbounded drift $A = i(\Delta - V)$ plays a crucial role.

Even for $x' = f_0(x) + \sum u_j(t) f_j(x)$ on a manifold M : no NSC for STGC
$\left(M \text{ connected}, \forall x_0 \in M, \text{Lie}\{f_1, \dots, f_m\}(x_0) = T_{x_0}M\right) \not\Rightarrow$ *STGC*

For PDEs : previous results

$$i\partial_t\psi(t,x) = \Big(-\Delta + V(x) + \sum_{j=1}^{m} u_j(t) W_j(x) \Big)\psi(t,x)\,, \qquad x \in M$$

- W_j bounded $\Rightarrow$ no exact control in $\mathcal{S}$ [Ball-Masden-Slemrod 1982]

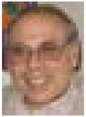
$\rightarrow$ We study the **Small-Time Global Approximate Controllability :**
$\forall \epsilon > 0, \psi_0, \psi_* \in \mathcal{S}, \quad \exists T \in [0,\epsilon], u \in L^1 \quad \text{s.t.} \quad \|\psi(T;u,\psi_0) - \psi_*\|_{L^2} < \epsilon$

- Large Time GAC is generic wrt W_j when $\text{Sp}(-\Delta + V)$ is discrete
[Boscain-Chambrion-Mason-Sigalotti 2009, Nersesyan 2010]

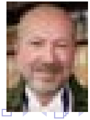
- Obstructions to STAC for some LTAC syst (harm. osc, gaussians)
[Mirahimi-Rouchon 2004, KB-Coron-Teismann 2014]

- **A key small-time result :** STC of phases [Duca-Nersesyan 2022]
$\forall \epsilon > 0, \psi_0 \in \mathcal{S}, \phi \in L^2, \exists T \in [0,\epsilon], u \in L^1, \|\psi(T;u,\psi_0) - \psi_0 e^{i\phi}\|_{L^2} < \epsilon$









Framework : Small-Time Approx. Reachable operators

$$i\partial_t\psi(t,x) = \Big(-\Delta + V(x) + \sum_{j=1}^{m} u_j(t) W_j(x) \Big)\psi(t,x)\,, \qquad x \in M\,.$$

• A unitary operator L on $L^2(M,\mathbb{C})$ is **STAR** if
$\forall \epsilon > 0, \psi_0 \in \mathcal{S} \quad \exists T \in [0,\epsilon], u \in L^1, \quad \|\psi(T;u,\psi_0) - L\psi_0\|_{L^2} < \epsilon.$

They form a semi-group closed for the strong convergence :
L_1 and L_2 STAR $\quad \Rightarrow \quad L_2 L_1$ STAR
$(L_n)_{n\in\mathbb{N}}$ STAR and $\forall \psi \in L^2(M), \|(L_n - L)\psi\|_{L^2} \xrightarrow[n\to\infty]{} 0 \quad \Rightarrow \quad L$ STAR

• The set of skew hermitian op H on $L^2(M)$ s.t. e^{tH} is STAR $\forall t \in \mathbb{R}$
has a Lie algebra structure $\star\star$ **if limits/commutators exist** $\star\star$

$$\left(e^{\frac{t}{n}H} e^{\frac{\widetilde{t}}{n}\widetilde{H}}\right)^n \xrightarrow[n\to\infty]{} e^{tH+\widetilde{t}\widetilde{H}} \qquad \left(e^{\frac{H}{n}} e^{\frac{\widetilde{H}}{n}} e^{-\frac{H}{n}} e^{-\frac{\widetilde{H}}{n}}\right)^{n^2} \xrightarrow[n\to\infty]{} e^{[H,\widetilde{H}]}$$

Ex : iW_j is STAR because $\forall a \in \mathbb{R}$, $e^{i\tau(\Delta-V)+iaW_j} \longrightarrow e^{iaW_j}$ as $\tau \to 0$.
Instantaneous phase change for ψ. How to go further ?

Proof of STC of phases : a key conjugation formula

For $x' = Ax + uBx$, the operator $e^{\frac{B}{\sqrt{\tau}}} e^{\tau A} e^{-\frac{B}{\sqrt{\tau}}}$ is approx. reach. in time τ^+ and $= \exp\left(\tau A + \sqrt{\tau}[B,A] + \frac{1}{2}\mathrm{ad}_B^2(A)\right)$ when $\mathrm{ad}_B^3(A) = 0$.

For $A = i(\Delta - V(x))$ and $B = iW(x)$ then

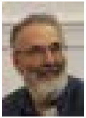
- $[B,A] = 2\nabla W \cdot \nabla + \Delta W$ is a transport operator
- $\mathrm{ad}_B^2(A) = -2i|\nabla W(x)|^2$
- $e^{\frac{B}{\sqrt{\tau}}} e^{\tau A} e^{-\frac{B}{\sqrt{\tau}}} \underset{\tau \to 0}{\longrightarrow} e^{-i|\nabla W(x)|^2}$: STAR unitary op, new phase chg !

Iteration + density $\Rightarrow$ $\forall \phi \in L^2(M)$, $e^{i\phi}$ is STAR.
This proves the **STGAC of phase** for

$$i\partial_t \psi = \Big(-\Delta + V(x) + \sum_{j=1}^{2d} u_j(t) W_j(x) \Big)\psi, \quad x \in M$$

- $M = \mathbb{T}^d,\ m = 2d+1,\ W_{2j-1} = \sin(x_j),\ W_{2j} = \cos(x_j)$ (trigo pol)
- $M = \mathbb{R}^d,\ m = d+2,\ W_j = x_j,\ W_{d+1} = e^{-|x|^2}$ (Hermite fns)

"saturation argument"
[Agrachev-Sarychev 2005]
[Duca-Nersesyan 2022]




Control separately angular and radial parts

Using polar decompositions $\psi_0 = \rho_0 e^{i\phi_0}$ and $\psi_* = \rho_* e^{i\phi_*}$

$$\psi_0 \xrightarrow{\text{STC of phase}} \rho_0 \xrightarrow{???} \rho_* \xrightarrow{\text{STC phase}} \psi_*$$

Are transport operators along gradient vector fields sufficient ?

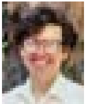
• The group of diffeomorphisms acts transitively on densities :
$\forall \rho_0, \rho_1 \in C^\infty(\mathbb{T}^d, \mathbb{R}_+^*) \cap \mathcal{S}$, $\exists P \in \mathrm{Diff}^0(\mathbb{T}^d)$ s.t. $\rho_1 = |\det(DP)|^{1/2} \rho_0 \circ P$
[Moser 1965]

• Diffeomorphisms on connected manifolds are compositions of flows.
[Thurston 1974]

• The vector fields whose flows are STAR $\forall t$ form a Lie algebra.

• The Lie algebra of gradient vector fields is "dense".

flows of grad vector fields $\Rightarrow$ any flow $\Rightarrow$ any diffeomorphism



How to move along flows of gradient vector fields ?

For $A = i(\Delta - V(x))$ and $B = iW(x)$ then
- $[B, A] = 2\nabla W \cdot \nabla + \Delta W$ is a transport operator
- $\mathrm{ad}_B^2(A) = -2i|\nabla W(x)|^2$

For $x' = Ax + uBx$, the operator $e^{\frac{B}{\tau}} e^{\tau A} e^{-\frac{B}{\tau}}$ is approx. reach. in time τ^+ and $= \exp\big(\tau A + [B, A] + \frac{1}{2\tau}\mathrm{ad}_B^2(A)\big)$ when $\mathrm{ad}_B^3(A) = 0$.

We remove the last term via Trotter-Kato formula : the operator

$$\left(e^{-\frac{1}{2n\tau}\mathrm{ad}_B^2(A)} e^{\frac{\tau}{n}A + \frac{1}{n}[B,A] + \frac{1}{2n\tau}\mathrm{ad}_B^2(A)}\right)^n \quad = \exp\left(\frac{\tau}{n}A + \frac{1}{n}[B, A] + O\left(\frac{1}{n^2}\right)\right)^n \xrightarrow{n\to\infty} e^{\tau A + [B,A]}.$$

is approximately reachable in time τ^+.

Using $\tau \to 0$, the operator $e^{[B,A]}$ is STAR.

Link with semi-classics and extensions

$$i\partial_t\psi = \big(-\Delta + V(x)\big)\psi, \quad x \in M$$

- **WKB approximation :** $\psi(sh;\psi_0 e^{i\frac{W}{h}}) = e^{i\frac{\phi(s,\cdot)}{h}}a(s,\cdot)$ where

$$\left| \begin{array}{l} \partial_s\phi(s,x) + |\nabla\phi(s,x)|^2 = 0, \\ \phi(0,\cdot) = W \end{array}\right. \qquad \left| \begin{array}{l} \big(\partial_s - ih(\Delta - V)a + \mathcal{T}_{\nabla\phi(s)}\big)\, a = 0 \\ a(0,\cdot) = \psi_0 \end{array}\right.$$

Eikonal equation $\qquad \mathcal{T}_f = \langle f, \nabla\cdot\rangle + \frac{1}{2}\mathrm{div}(f)$

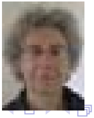
For instance, with $A = i(\Delta - V)$ and $B = -iW$ and $h = \sqrt{\tau}$ then

$$e^{\frac{B}{\sqrt{\tau}}} e^{\tau A} e^{-\frac{B}{\sqrt{\tau}}}\psi_0 = e^{-i\frac{W-\phi(\sqrt{\tau},\cdot)}{\sqrt{\tau}}} a(\sqrt{\tau},\cdot) \quad \underset{\tau\to 0}{\longrightarrow} \quad e^{i|\nabla W|^2}\psi_0$$

The control of diffeomorphisms makes use of the transport term.

- Extension to log-NLS [KB, Carles, Pozzoli 2026]

$$i\partial_t\psi = \big(-\Delta + V(x) + \lambda\psi\log|\psi|^2 + \sum u_j(t)W_j(x)\big)\psi, \quad x \in M$$



Conclusion, open problems

- A new method to prove global L^2-approximate controllability for
$$i\partial_t\psi = \Big(-\Delta + V(x) + \sum_{j=1}^{2d} u_j(t) W_j(x) \Big)\psi, \quad x \in M$$
combining control of phase and diffeomorphisms.

- It applies in 2 cases :
  - $M = \mathbb{T}^d,\ m = 2d+1,\ W_{2j-1} = \sin(x_j),\ W_{2j} = \cos(x_j)$,
  - $M = \mathbb{R}^d,\ m = d+2,\ W_j = x_j,\ W_{d+1} = e^{-|x|^2}$.

- It works in **arbitrary small time**, with **continuous spectrum** and log-nonlinearities.

- It is based on well-defined **Lie brackets along which one can move in small time.**

- H^s-approximate controllability, $s > 0$?
  Polar decomposition is irregular !

- Genericity wrt $V, W_1, \dots, W_m$?

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